

ON A CLASS OF SPECIAL RIEMANNIAN MANIFOLD

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ABSTRACT. We consider a 4-dimensional Riemannian manifold M with a metric g and an affinor structure q . We note the local coordinates of g and q are circulant matrices. Their first orders are (A, B, C, B) , $A, B, C \in FM$ and $(0, 1, 0, 0)$, respectively.

Let ∇ be the connection of g . Then we obtain:

- 1) $q^4 = E$; $g(qx, qy) = g(x, y)$, $x, y \in \chi M$,
- 2) $\nabla q = 0$ if and only if $\text{grad}A = (\text{grad}C)q^2$; $2\text{grad}B = (\text{grad}C)(q + q^3)$,

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1. INTRODUCTION

The main purpose of the present paper is to find a class of Riemannian manifolds which admits a circulant metric g , as well as an additional circulant structure q , such that $q^4 = id$, and q is a parallel structure with respect to the Riemannian connection ∇ of g .

2. PRELIMINARIES

We consider a 4-dimensional Riemannian manifold M with a metric g and an affinor structure q . We note the local coordinates of g and q are circulant matrices. The analogous case of a three dimensional Riemannian manifold has been discussed in [1], [2], [3].

So let the metric g have the coordinates:

$$(1) \quad g_{ij} = \begin{pmatrix} A & B & C & B \\ B & A & B & C \\ C & B & A & B \\ B & C & B & A \end{pmatrix}, \quad \det g_{ij} = (A - C)^2((A + C)^2 - 4B^2)$$

in the local coordinate system (x_1, x_2, x_3, x_4) , and $A = A(p)$, $B = B(p)$, $C = C(p)$, where $p(x_1, x_2, x_3, x_4) \in F \subset R^4$. Naturally, A, B, C are smooth functions of a point p . We suppose $A > C > B > 0$. These conditions imply the heat minors of the matrix g are positive, so the metric g is positively defined [5]. The inverse matrix is

$$(2) \quad g^{ij} = \frac{1}{D} \begin{pmatrix} \overline{A} & \overline{B} & \overline{C} & \overline{B} \\ \overline{B} & \overline{A} & \overline{B} & \overline{C} \\ \overline{C} & \overline{B} & \overline{A} & \overline{B} \\ \overline{B} & \overline{C} & \overline{B} & \overline{A} \end{pmatrix}, \quad D = (A - C)((A + C)^2 - 4B^2),$$

where $\overline{A} = A(A + C) - 2B^2$, $\overline{B} = B(C - A)$, $\overline{C} = 2B^2 - C(A + C)$.

Further, let the local coordinates of q be

$$(3) \quad q_i^j = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

We will use the notation $\Phi_i = \frac{\partial \Phi}{\partial x^i}$ for every smooth function Φ defined in F .

3. THE CONDITION FOR A PARALLEL STRUCTURE

Theorem 3.1. *Let M be a 4-dimensional Riemannian manifold with a metric g and an affinor structure q with local coordinates (1), (3), respectively. Then we have*

$$(4) \quad q^4 = E; \quad q^2 \neq \pm E$$

$$(5) \quad g(qu, qv) = g(u, v), \quad u, v \in \chi M,$$

where E is the unit matrix.

Proof. The conditions (4) follows directly from (3). \square

Now let $u = (u^1, u^2, u^3, u^4)$ and $v = (v^1, v^2, v^3, v^4)$ be two vectors in χM . Using (1) and (3) we calculate that

$$g(u, u) = A((u^1)^2 + (u^2)^2 + (u^3)^2 + (u^4)^2) + 2B(u^1u^2 + u^1u^4 + u^2u^3 + u^3u^4) + 2C(u^1u^3 + u^2u^4),$$

$g(qu, qu) = g(u, u)$. It is easily to see

$$g(q^3u, q^3u) = g(q^2u, q^2u) = g(qu, qu) = g(u, u),$$

$$g(q^3u, q^3v) = g(q^2u, q^2v) = g(qu, qv) = g(u, v).$$

Proposition 3.2. *Let $u = (u^1, u^2, u^3, u^4)$ and $v = (v^1, v^2, v^3, v^4)$ be two vectors in χM , then $g(q^3u, q^3v) = g(q^2u, q^2v) = g(qu, qv) = g(u, v)$.*

Theorem 3.3. *Let M be a Riemannian manifold with a metric g from (1) and affinor structure from (3). Let ∇ be the Riemannian connection of g . Then $\nabla q = 0$ if and only if, when*

$$(6) \quad \text{grad}A = (\text{grad}C)q^2; \quad 2\text{grad}B = (\text{grad}C)(q + q^3).$$

Proof. Let Γ_{ij}^s be the Christoffel symbols of ∇ . Let $\nabla q = 0$. That means

$$(7) \quad \nabla_i q_j^s = \partial_i q_j^s + \Gamma_{ik}^s q_j^k - \Gamma_{ij}^k q_k^s = 0$$

From (3) and (7) we get

$$(8) \quad \Gamma_{ik}^s q_j^k = \Gamma_{ij}^k q_k^s$$

Using (1), (2), (3), (4), (8) and the well known identities:

$$(9) \quad 2\Gamma_{ij}^s = g^{as}(\partial_i g_{aj} + \partial_j g_{ai} - \partial_a g_{ij}).$$

After a long computation we get the following system:

$$\begin{aligned}
A_4 - B_1 + B_3 - C_2 &= 0, \\
A_4 + B_1 - B_3 - C_2 &= 0, \\
2A_2 + A_4 - 3B_1 - B_3 + C_2 &= 0, \\
A_3 + B_2 - B_4 - C_1 &= 0, \\
A_3 - B_2 + B_4 - C_1 &= 0, \\
A_2 - B_1 + B_3 - C_4 &= 0, \\
A_2 + B_1 - B_3 - C_4 &= 0, \\
A_4 - B_1 + 3B_3 + C_2 + 2C_4 &= 0, \\
A_2 + 2A_4 - 3B_1 - B_3 + C_4 &= 0, \\
A_2 + 2A_4 - B_1 - 3B_3 + C_4 &= 0, \\
A_1 + 2A_3 - 3B_2 - B_4 + C_3 &= 0, \\
A_1 - B_2 + B_4 - C_3 &= 0, \\
A_3 - B_2 - 3B_4 + C_1 + 2C_3 &= 0, \\
A_1 - B_2 - 3B_4 + 2C_1 + C_3 &= 0, \\
2A_1 + A_3 - B_2 - 3B_4 + C_1 &= 0, \\
A_2 - B_1 - 3B_3 + 2C_2 + C_4 &= 0.
\end{aligned}$$

The last system implies:

$$\begin{aligned}
A_1 &= C_3, \quad A_2 = C_4, \quad A_3 = C_1, \quad A_4 = C_2, \quad B_1 = B_3 \\
(10) \quad B_2 &= B_4, \quad 2B_1 = C_4 + C_2, \quad 2B_2 = C_1 + C_3.
\end{aligned}$$

From (10) we find that (6) is valid.

Inversely, let (6) be valid. We can verify that (10) is valid, too. The identities (10) imply (8) and consequently (7) is true. So $\nabla q = 0$. \square

Note. In fact (6) is a system of partial differential equations for the functions A , B and C . We can say that (6) has a solution.

Now, we will give an example of such a manifold. Let

$$\begin{aligned}
(11) \quad A &= (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2, \\
B &= x^1 x^2 + x^2 x^3 + x^1 x^4 + x^3 x^4, \\
C &= 2x^1 x^3 + 2x^2 x^4
\end{aligned}$$

be three functions of a point $p(x^1, x^2, x^3, x^4) \neq (x, x, x, x)$, $p(x^1, x^2, x^3, x^4) \neq (-x, x, -x, x)$. Then $A > C > B > 0$ and

$$(12) \quad g_{ij} = \begin{pmatrix} A & B & C & B \\ B & A & B & C \\ C & B & A & B \\ B & C & B & A \end{pmatrix}$$

is positively defined. Also, we obtain $\text{grad} A = (\text{grad} C)q^2$; $2\text{grad} B = (\text{grad} C)(q + q^3)$, which implies $\nabla q = 0$. So, we find an example for a manifold M with a metric g , defined by (11) and (12), and affinor structures q , defined by (3), which satisfies $\nabla q = 0$.

Let R be the curvature tensor field of ∇ , i.e $R(x, y)z = \nabla_x \nabla_y z - \nabla_{[x, y]}z$. We consider the associated with R tensor field R of type $(0, 4)$, defined by the condition

$$R(x, y, z, u) = g(R(x, y)z, u), \quad x, y, z, u \in \chi M.$$

Theorem 3.4. *If M is the Riemannian manifold with a metric g and a parallel structure q , defined by (1) and (3), respectively, then the curvature tensor R of g satisfies the identity:*

$$(13) \quad R(x, y, z, qu) = R(x, y, q^3 z, u), \quad x, y, z, u \in \chi M.$$

Proof. In the terms of the local coordinates $\nabla q = 0$ implies

$$(14) \quad R_{sji}^l q_k^s = R_{kji}^s q_s^l.$$

Using (3) we can verify $q_j^i = q_a^t q_j^a q_t^i$ and then from (1), (3) and (14) we obtain (13). \square

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